

On Critical Independent Sets of a Graph and Structure of Unicyclic Non-König-Egerváry Graphs

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Abstract

Let G be a finite simple graph. For $X \subset V(G)$, the *difference* of X , $d(X) := |X| - |N(X)|$ where $N(X)$ is the neighbourhood of X and $\max \{d(X) : X \subset V(G)\}$ is called the *critical difference* of G . X is called a *critical set* if $d(X)$ equals the critical difference and $\text{Ker}(G)$ is the intersection of all critical sets. It is known that $\text{Ker}(G)$ is an independent (vertex) set of G . $\text{Diadem}(G)$ is the union of all critical independent sets. An independent set S is an *inclusion minimal set with* $d(S) > 0$ if no proper subset of S has positive difference.

A graph G is called *König-Egerváry* if the sum of its independence number ($\alpha(G)$) and matching number ($\mu(G)$) equals the order of G . It is known that bipartite graphs are König-Egerváry.

In this paper, we study independent sets with positive difference for which every proper subset has a smaller difference and prove a result conjectured by Levit and Mandrescu in 2013. The conjecture states that for any graph, the number of inclusion minimal sets S with $d(S) > 0$ is at least the critical difference of the graph. We also give a short proof of the inequality $|\text{Ker}(G)| + |\text{Diadem}(G)| \leq 2\alpha(G)$ (proved by Short in 2016).

A characterisation of unicyclic non-König-Egerváry graphs is also presented and a conjecture which states that for such a graph G , the critical difference equals $\alpha(G) - \mu(G)$, is proved.

We also make an observation about $\text{Ker}(G)$ using Edmonds-Gallai Structure Theorem as a concluding remark.

Mathematics Subject Classifications: 05C69, 05C70, 05A20

1 Introduction

In this paper G is a finite simple graph with the finite vertex set $V(G)$ and the edge set $E(G) \subset \binom{V(G)}{2}$. For $A \subset V(G)$, neighbourhood of A , $N(A)$ is defined to be the set of all vertices adjacent to some vertex in A . The *degree* of a vertex v is the number of edges incident to v and is denoted by $\deg(v)$. A set S of vertices is *independent* if no two vertices from S are adjacent. An independent set of maximum size is a *maximum independent set* of G and its cardinality is denoted by $\alpha(G)$. The reader may refer to any of the standard text books [2, 11] for basic notations.

This paper is based on the results in [3, 4, 5, 6, 7, 8, 9, 10, 12, 14]. To state the results proved in this paper some more definitions are included.

Let $\text{Ind}(G) = \{S : S \text{ is an independent set of } G\}$, $\Omega(G) = \{S \in \text{Ind}(G) : |S| = \alpha(G)\}$ and $\text{Core}(G) = \cap \{S : S \in \Omega(G)\}$ [5].

$d_G(X) = |X| - |N(X)|$, $X \subset V(G)$, is called the *difference* of the set X . When there is no confusion the the subscript G may be dropped. The number

$$d_c(G) = \max \{d(X) : X \subset V(G)\},$$

is called the *critical difference* of G . A set $U \subset V(G)$ is *critical* if $d(U) = d_c(G)$ [14] and $\text{Ker}(G)$ is the intersection of all critical sets [7]. *Diadem* of a graph is defined as the union of all critical independent sets and it is denoted by $\text{Diadem}(G)$ [3]. One may observe that $\text{Ker}(G) \in \text{Ind}(G)$. An independent set S is called inclusion minimal set with $d(S) > 0$ if no proper subset of S has positive difference [8]. In this paper by “inclusion minimal” we will also imply $d(S) > 0$ and not state it explicitly every time.

A matching in a graph G is a set M of edges such that no two edges in M share a common vertex. Size of a largest possible matching (*maximum matching*) is denoted by $\mu(G)$. A vertex is *matched* (or *saturated*) by M if it is an endpoint of one of the edges in M . A perfect matching is a matching which matches all vertices of the graph. For two disjoint sets $A, B \subset V(G)$ we say there is a *matching from A into B* if there is a matching M such that any edge in M joins a vertex in A and a vertex in B and all the vertices in A are matched by M .

A graph G is a König-Egerváry (KE) graph if $\alpha(G) + \mu(G) = |V(G)|$ [1, 13]. König-Egerváry graphs have been well studied. Levit and Mandrescu studied the critical difference, Ker , Core , Diadem of graphs, properties of König-Egerváry graphs and proved several results. Based on these results several natural conjectures and problems arose. The ones considered in this paper are stated below.

Conjecture 1.1. [8] *For any graph G , the number of inclusion minimal independent set S such that $d(S) > 0$ is at least $d_c(G)$.*

Theorem 1.2. *For any graph G , $|\text{Ker}(G)| + |\text{Diadem}(G)| \leq 2\alpha(G)$.¹*

Conjecture 1.3. *For a unicyclic non-KE graph G , $d_c(G) = \alpha(G) - \mu(G)$.²*

Problem 1.4. [3, 9] *Characterise graphs such that $\text{Core}(G)$ is critical.*

¹Conjectured in [3] and proved in [12].

²Stated by Levit in a talk at TIFR in 2014.

Problem 1.5. [3, 9] *Characterise graphs with $\text{Ker}(G) = \text{Core}(G)$.*

In Section 2 Conjecture 1.1 and related results are proved. In Section 3 a short proof of Theorem 1.2 is given. A characterisation of unicyclic non-König-Egerváry graph is presented in Section 4 and as a corollary Conjecture 1.3 is proved. In the concluding section Edmonds-Gallai structure Theorem is used to make an observation regarding $\text{Ker}(G)$. It may be useful for Problems 1.4 and 1.5.

2 On Minimum Number of Inclusion Minimal Sets

In this section we study independent sets X with $d(X) > 0$ and $d(Y) < d(X)$ for all $Y \subsetneq X$ and give a proof of Conjecture 1.1. There are several results that led to the formulation of this conjecture. Some of them are listed below as they help to understand the proof or they are used in the proof of the conjecture.

Theorem 2.1. [4] *There is a matching from $N(S)$ into S for every critical independent set S .*

Theorem 2.2. [7] *For every graph G , the following assertions are true:*

1. $\text{Ker}(G)$ is the unique minimal critical independent set of G .
2. $\text{Ker}(G) \subset \text{Core}(G)$.

Theorem 2.3. [7] *For a graph G , the following assertions are true:*

1. The function d is supermodular, i.e., $d(X \cup Y) + d(X \cap Y) \geq d(X) + d(Y)$ for every $X, Y \subset V(G)$.
2. If X and Y are critical in G , then $X \cup Y$ and $X \cap Y$ are critical as well.

Theorem 2.4. [9] *If G is a bipartite graph, then $\text{Ker}(G) = \text{Core}(G)$.*

Theorem 2.5. [8] *For a vertex v in a graph G , the following assertions hold:*

1. $d(G - v) = d(G) - 1$ if and only if $v \in \text{Ker}(G)$;
2. if $v \in \text{Ker}(G)$, then $\text{Ker}(G - v) \subset \text{Ker}(G) - v$.

Theorem 2.6. [8] *If $\text{Ker}(G) \neq \emptyset$, then*

$\text{Ker}(G) = \cup \{S : S \text{ is an inclusion minimal independent set with } d(S) > 0\}$

For an independent set X of G a new graph H_X is defined as follows. The vertex set $V(H_X) = X \cup N(X) \cup \{v, w\}$, where v and w are two new vertices not in $V(G)$ and the edge set $E(H_X) = \{xy \in E(G) : x \in X, y \in N(X)\} \cup \{vw\} \cup \{vx : x \in N(X)\}$. Note that H_X is a connected bipartite graph. Figure 1 gives an illustration of the construction. Also observe that $d_{H_X}(Y) = d_G(Y)$ for all $Y \subset X$.

Theorem 2.7. *If X is an independent set of G with $d(X) > 0$ and $d(Y) < d(X)$ for all $Y \subsetneq X$ then $\text{Ker}(H_X) = X$.*

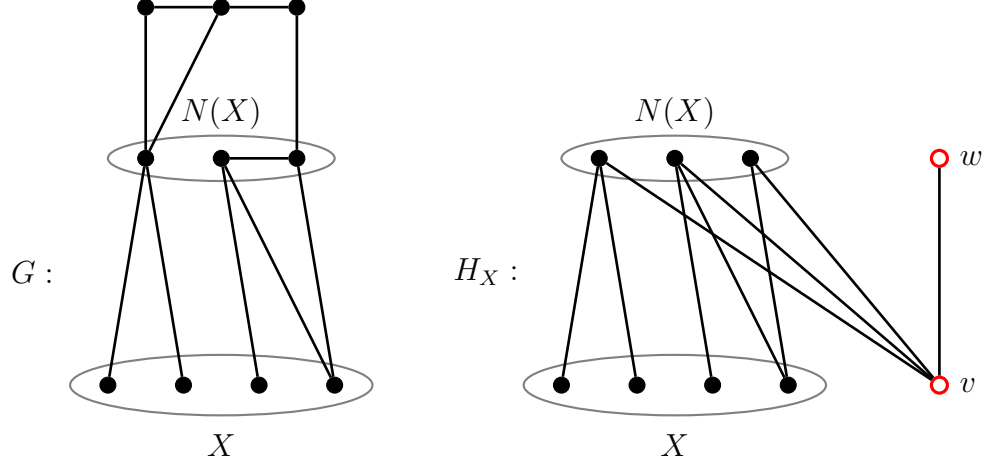


Figure 1: Construction of H_X for a given independent set X of G .

Proof. It may be noted that the only maximal independent set that contains v is $X \cup v$ and its size is $|X| + 1$. If a maximal independent set does not contain v , it contains w . Next it is shown that size of a largest independent set that contains w is also $|X| + 1$.

Let $I = S \cup T \cup w$ be a maximum independent set where $S \subset X$ and $T \subset N(X)$.

$$\begin{aligned}
 & |N(S) \cup T| &= & |N(X)| \text{ (since } I \text{ is maximal independent)} \\
 \Rightarrow & |N(S)| + |T| &= & |N(X)| \text{ (since } N(S) \text{ and } T \text{ are disjoint)} \\
 \Rightarrow & |N(S)| + |X - S| &\leq & |N(X)| \text{ (otherwise } |X \cup w| > |I|) \\
 \Rightarrow & |X| - |N(X)| &\leq & |S| - |N(S)| \\
 \Rightarrow & d(X) &\leq & d(S), \text{ a contradiction if } S \subsetneq X.
 \end{aligned}$$

Thus, $I = X \cup w$. Hence, $X \cup v$ and $X \cup w$ are the only two maximum independent sets of H_X . So $\text{Core}(H_X) = (X \cup v) \cap (X \cup w) = X$. Since H_X is bipartite, Theorem 2.4 implies $\text{Ker}(H_X) = \text{Core}(H_X) = X$. \square

Corollary 2.8. *If X is an independent set of G with $d(X) > 0$ and $d(Y) < d(X)$ for all $Y \subsetneq X$, then X can be expressed as a union of inclusion minimal sets S with $d(S) > 0$.*

Proof. By Theorem 2.6, $X = \text{Ker}(H_X)$ is the union of all inclusion minimal sets of H_X . As inclusion minimal sets of H_X are contained in X , they are inclusion minimal sets of G . Hence, the result follows. \square

Corollary 2.9. *If X is an independent set of G with $d(X) > 0$ and $d(Y) < d(X)$ for all $Y \subsetneq X$, then $X \subset \text{Ker}(G)$.*

Proof. From Corollary 2.8 it follows X is contained in the union of inclusion minimal subsets of G . From Theorem 2.6 it follows that X is contained in $\text{Ker}(G)$. \square

Corollary 2.10. *If S is an inclusion minimal set of a graph G with $d_G(S) > 0$, then $d_G(S) = 1$.*

Proof. This result was first shown in [8]. From Theorem 2.7, we have $\text{Ker}(H_S) = S$. By Theorem 2.5,

$$d_{H_S}(S - v) = d_c(H_S) - 1 = d_{H_S}(S) - 1 = d_G(S) - 1, \text{ for all } v \in S.$$

Thus, $d_G(S - v) = d_G(S) - 1$ for all $v \in S$. As S is an inclusion minimal set, $d_G(S - v) \leq 0$, which implies, $d_G(S) \leq 1$. Hence, $d_G(S) = 1$. \square

Corollary 2.8 can be made stronger and it can also be proved directly.

Theorem 2.11. *Let $X \in \text{Ind}(G)$ with $d(X) = k > 0$. If $d(Y) < k$ for all $Y \subsetneq X$, then X can be expressed as a union of k distinct inclusion minimal sets.*

Proof. We note that if $S \subset X$ then

$$\begin{aligned} d(X - S) &= |X - S| - |N(X - S)| \\ &= |X| - |S| - |N(X - S)| \\ &\geq |X| - |N(X)| - |S| \\ &= d(X) - |S| \\ &= k - |S|. \end{aligned}$$

It may be verified that if $d(S) > 0$, then S contains an inclusion minimal set T with $d(T) > 0$.

Since, $d(X) > 0$, X contains an inclusion minimal set. Choose an inclusion minimal set $S_1 \subset X$ and a vertex $s_1 \in S_1$. Suppose $(s_1, S_1), (s_2, S_2), \dots, (s_i, S_i)$, where $i < k$ have been selected. As, $d(X - \{s_1, s_2, \dots, s_i\}) \geq k - i > 0$, $X - \{s_1, s_2, \dots, s_i\}$ contains an inclusion minimal set. Choose an inclusion minimal set S_{i+1} in $X - \{s_1, s_2, \dots, s_i\}$ and a vertex s_{i+1} in S_{i+1} . This process is continued till the pairs $(s_1, S_1), (s_2, S_2), \dots, (s_k, S_k)$ are obtained. Note that, $s_i \notin S_j$, for $1 \leq i < j \leq k$.

Next it is shown that $d(S_1 \cup S_2 \cup \dots \cup S_k) \geq k$. This is proved by repeated application of supermodularity of d (Theorem 2.3).

By supermodularity of d , for $i \in \{1, 2, \dots, k-1\}$,

$$\begin{aligned} d(S_k \cup S_{k-1} \cup \dots \cup S_i) + d((S_k \cup S_{k-1} \cup \dots \cup S_{i+1}) \cap S_i) &\geq \\ d(S_k \cup S_{k-1} \cup \dots \cup S_{i+1}) + d(S_i). \end{aligned} \tag{1}$$

Now $s_{k-1} \in S_{k-1} - S_k$ implies $S_k \cap S_{k-1} \subsetneq S_{k-1}$. Since S_{k-1} is inclusion minimal and $S_k \cap S_{k-1} \subsetneq S_{k-1}$, $d(S_k \cap S_{k-1}) \leq 0$ holds. Also, $d(S_k) = d(S_{k-1}) = 1$. Then Equation (1) for $i = k-1$ implies $d(S_k \cup S_{k-1}) \geq 2$.

In general, for $i \in \{1, 2, \dots, k-1\}$, $s_i \in S_i - \bigcup_{j=i+1}^k S_j$. Hence $(S_k \cup S_{k-1} \cup \dots \cup S_{i+1}) \cap S_i \subsetneq S_i$, implying $d((S_k \cup S_{k-1} \cup \dots \cup S_{i+1}) \cap S_i) \leq 0$.

If $d(S_k \cup S_{k-1} \cup \dots \cup S_{i+1}) \geq k-i$, then Equation (1) implies $d(S_k \cup S_{k-1} \cup \dots \cup S_i) \geq k-i+1$. In particular $d(S_k \cup S_{k-1} \cup \dots \cup S_1) \geq k$. This leads to a contradiction if $\bigcup_{i=1}^k S_i \subsetneq X$. Thus $X = \bigcup_{i=1}^k S_i$. \square

Conjecture 1.1 follows as an immediate consequence of this theorem.

Corollary 2.12. *The number of inclusion minimal independent sets of G is greater than or equal to $d_c(G)$.*

Proof. As $\text{Ker}(G)$ is the unique minimal critical independent set of G , setting $X = \text{Ker}(G)$ in Theorem 2.11 this corollary and Conjecture 1.1 is proved. \square

Converse of Corollary 2.8 also holds true.

Theorem 2.13. *If $X \subset V(G)$ can be expressed as a union of inclusion minimal sets with positive difference, then $X \in \text{Ind}(G)$, $d(X) > 0$ and $d(Y) < d(X)$ for all $Y \subsetneq X$.*

Proof. Let $X = \cup_{i=1}^k S_i$ where S_i is an inclusion minimal set with $d(S_i) > 0$ for all $i \in [k]$. Since X is contained in the union of inclusion minimal sets, $X \subset \text{Ker}(G)$ and thus $X \in \text{Ind}(G)$.

Let $Y \subsetneq X$. Then at least one of the S_i 's is not contained in Y . Let, without loss of generality, $S_1 \not\subset Y$. Now, for $S_0 \subset X$ and $i = 2, 3, \dots, k$, by supermodularity of d ,

$$\begin{aligned} d(S_0 \cup S_1 \cup \dots \cup S_i) + d((S_0 \cup S_1 \cup \dots \cup S_{i-1}) \cap S_i) &\geq \\ d(S_0 \cup S_1 \cup \dots \cup S_{i-1}) + d(S_i). \end{aligned} \quad (2)$$

As, for $i = 2, 3, \dots, k$, S_i is inclusion minimal and $(S_0 \cup S_1 \cup \dots \cup S_{i-1}) \cap S_i \subset S_i$, $d(S_i) = 1$ and $d((S_0 \cup S_1 \cup \dots \cup S_{i-1}) \cap S_i) \leq 1$. Hence, (2) gives, $d(S_0 \cup S_1 \cup \dots \cup S_i) \geq d(S_0 \cup S_1 \cup \dots \cup S_{i-1})$ for $i = 2, 3, \dots, k$. This implies,

$$\begin{aligned} d(X) &= d(S_0 \cup S_1 \cup \dots \cup S_k) \\ &\geq d(S_0 \cup S_1 \cup \dots \cup S_{k-1}) \\ &\geq \dots \\ &\geq d(S_0 \cup S_1). \end{aligned} \quad (3)$$

Set $S_0 = \emptyset$ in (3) to get $d(X) \geq d(S_1) > 0$.

Set $S_0 = Y$ in (3) to get $d(X) \geq d(Y \cup S_1)$. Again by supermodularity of d ,

$$d(Y \cup S_1) + d(Y \cap S_1) \geq d(Y) + d(S_1).$$

As $S_1 \not\subset Y$, $Y \cap S_1 \subsetneq S_1$ and hence $d(Y \cap S_1) \leq 0$. Thus

$$d(Y \cup S_1) \geq d(Y) + d(S_1) > d(Y).$$

The result follows. \square

We state one more result on criticality of independent sets which will be used in Section (4). This is a converse of Theorem 2.1.

Theorem 2.14. *Let X be an independent set of a graph G containing $\text{Ker}(G)$. If there is a matching from $N(X)$ into X , then X is critical.*

Proof. Let $A = X - \text{Ker}(G)$ and $B = N(X) - N(\text{Ker}(G))$. Since there is a matching from $N(X)$ into X and there are no edges between B and $\text{Ker}(G)$, by Hall's Matching Theorem [2, 11],

$$|B| \leq |N(B) \cap X| \leq |X - \text{Ker}(G)| = |A|.$$

Then,

$$\begin{aligned} d(X) &= |X| - |N(X)| \\ &= (|\text{Ker}(G)| + |A|) - (|N(\text{Ker}(G))| + |B|) \\ &= d_c(G) + |A| - |B| \\ &\geq d_c(G). \end{aligned}$$

Thus $d(X) = d_c(G)$. □

Corollary 2.15. *Core(G) is critical if and only if there is a matching from $N(\text{Core}(G))$ into $\text{Core}(G)$.*

Proof. Follows from Theorem 2.1, Theorem 2.2 and Theorem 2.14. □

3 A Ker-Diadem Inequality

It was conjectured in [3] that the sum of sizes of Ker and Diadem is at most twice the independence number. Taylor Short proved this inequality in [12] using structural results by Larson in [4]. Here a short and direct proof for this “Ker-Diadem Inequality” is presented.

Lemma 3.1. *If X and Y are two critical independent sets of a graph G then $|N(X) \cap Y| = |N(Y) \cap X|$.*

Proof. Let $S = N(X) \cap Y$. As $S \subset Y$, $N(S) \subset N(Y)$, and hence $N(S) \cap X \subset N(Y) \cap X$. Using Hall's Theorem it can be verified that if X is a critical independent set then there is a matching from $N(X)$ into X (Theorem 2.1). As $S \subset N(X)$, by Hall's Theorem, $|S| \leq |N(S) \cap X| \leq |N(Y) \cap X|$, i.e., $|N(X) \cap Y| \leq |N(Y) \cap X|$. By a similar argument, $|N(Y) \cap X| \leq |N(X) \cap Y|$. □

Theorem 3.2. *If X is a maximal critical independent set of graph G then $\text{Diadem}(G) \subset X \cup N(X) - N(\text{Ker}(G))$.*

Proof. First it is shown that $\text{Diadem}(G) \subset X \cup N(X)$. Suppose $x \in \text{Diadem}(G) - (X \cup N(X))$. Then there exists a critical independent set A containing x . Define $Y = X \cup A - (N(X) \cap A)$. Observe that Y is an independent set and $X \subsetneq Y$. This implies $N(Y) \subset N(X \cup A) - (X \cap N(A))$ (in fact equality holds).

$$\begin{aligned} \text{Hence, } d(Y) &= |Y| - |N(Y)| \\ &\geq |X \cup A - (N(X) \cap A)| - |N(X \cup A) - (X \cap N(A))| \\ &= |X \cup A| - |N(X) \cap A| - |N(X \cup A)| + |X \cap N(A)| \\ &= |X \cup A| - |N(X \cup A)| \text{ (using Lemma 3.1)} \\ &= d(X \cup A). \end{aligned}$$

By supermodularity of d and criticality of X and A , it may be shown that $X \cup A$ is also critical (Theorem 2.3). Thus, $d(Y) = d_c(G)$ and this contradicts that X is a maximal critical independent set. This shows that $\text{Diadem}(G) \subset X \cup N(X)$.

For any critical independent set I , $I \cap N(\text{Ker}(G)) = \emptyset$. Thus taking union over all such I 's, we get $\text{Diadem}(G) \cap N(\text{Ker}(G)) = \emptyset$. Hence, $\text{Diadem}(G) \subset X \cup N(X) - N(\text{Ker}(G))$. \square

Corollary 3.3. *For every graph G , $|\text{Diadem}(G)| + |\text{Ker}(G)| \leq 2\alpha(G)$.*

Proof. Let X be a maximal critical independent set of G . By Theorem 3.2, $\text{Diadem}(G) \subset X \cup N(X) - N(\text{Ker}(G))$. Therefore,

$$\begin{aligned} |\text{Diadem}(G)| &\leq |X \cup N(X)| - |N(\text{Ker}(G))| \text{ (as } N(\text{Ker}(G)) \subset N(X)) \\ &= |X| + |N(X)| - |N(\text{Ker}(G))| \text{ (as } X \text{ is independent)} \\ &= |X| + (|X| - d_c(G)) - |N(\text{Ker}(G))| \text{ (as } X \text{ is critical)} \\ &= 2|X| - (d_c(G) + |N(\text{Ker}(G))|) \\ &= 2|X| - |\text{Ker}(G)| \\ &\leq 2\alpha(G) - |\text{Ker}(G)|. \end{aligned}$$

\square

4 Characterisation of Unicyclic non-KE Graphs

For any graph G , $\alpha(G) + \mu(G) \leq |V(G)|$ and it is well known that for bipartite graphs equality holds [1, 13]. It turned out that many interesting properties can be proved for graphs G which satisfy $\alpha(G) + \mu(G) = |V(G)|$. This motivated the study of graphs for which this equality holds. A graph is called König-Egerváry (KE) if $\alpha(G) + \mu(G) = |V(G)|$. KE graphs have been studied extensively [3]. This motivated researchers to consider graphs that are close to KE graphs. One of these classes is unicyclic graphs. For any unicyclic graph G , $|V(G)| - 1 \leq \alpha(G) + \mu(G) \leq |V(G)|$.

In this section we characterise the unicyclic non-KE graphs and prove some properties. These properties also lead to a proof of the Conjecture 1.3.

We present a procedure to construct any connected non-KE graph G .

Procedure 4.1. *A connected graph G is constructed by the following Steps.*

1. *Construct an odd cycle, colour its vertices blue.*
2. *Go to Step (3) or Step (4) or stop.*
3. *Attach a path of length two to any vertex: Choose a vertex u and then add two new vertices u_1 and u_2 and two edges uu_1 and u_1u_2 . Color u_1 red and u_2 black. Go to Step (2).*
4. *Attach a black leaf to a red vertex: Choose a vertex u which is coloured red and add a new vertex u_1 and an edge uu_1 . Color u_1 black. Go to Step (2).*

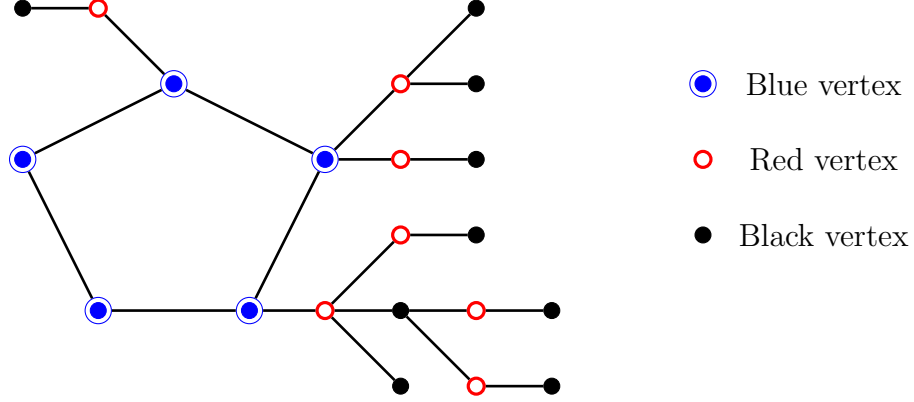


Figure 2: A graph constructed by Procedure 4.1.

Let G be a graph constructed by Procedure 4.1. Henceforth the unique cycle will be denoted by C . $G - E(C)$ is a forest. Consider the components of this forest as rooted trees with the root being the blue vertex of C . The notions of *parent*, *child*, *descendant* etc. are used with respect to these rooted trees. It may be noted that leaves are always coloured black. See Figure 2 for an example of such graph.

Lemma 4.2. *If the graph G constructed by Procedure 4.1 is unicyclic non-KE, then after applying Step 3 the new graph G' is also unicyclic non-KE.*

Proof. We may add the new edge u_1u_2 to any matching M in G and the vertex u_2 to any independent set I of G to get a new matching M' and an independent set I' in G' . This implies $\mu(G') + \alpha(G') \geq \mu(G) + \alpha(G) + 2$

Since the added two extra vertices are adjacent, $\alpha(G') \leq \alpha(G) + 1$. Also the two added edges have a vertex in common. So $\mu(G') \leq \mu(G) + 1$. This yields $\mu(G') + \alpha(G') \leq \mu(G) + \alpha(G) + 2$. Thus, $\mu(G') + \alpha(G') = |V(G')| - 1$. \square

Theorem 4.3. *There exists a maximum independent set that contains all the black vertices of a graph G constructed by Procedure 4.1.*

Proof. Note that black vertices are added one at a time in Step (3) or Step (4) of Procedure 4.1. This theorem is proved by induction on the number of black vertices.

Base Case: If there are no black vertices the result is trivially true.

Induction Hypothesis: There exists a maximum independent set that contains all the black vertices when the number of black vertices is $\ell - 1$.

Induction Step: Number of black vertices is ℓ . Look at the most recently added black vertex in G by Step (3) or (4). Call it w and its parent (which is a red vertex) v .

Case 1: w is added by Step (3).

$G' = G - v - w$ can be constructed by Procedure 4.1 and the number of black vertices in G' is $\ell - 1$. By induction hypothesis, there exists a maximum independent set I' of G' containing all the black vertices. Then $I = I' \cup w$, is a maximum independent set of G containing all the black vertices of G .

Case 2: w is added by Step (4).

$G' = G - w$ can be constructed by Procedure 4.1 and the number of black vertices in G' is $\ell - 1$. By induction hypothesis, there exists a maximum independent set I' of G' containing all the black vertices. The parent v of w has at least one black child other than w . This implies $v \notin I'$. Hence, $I = I' \cup w$ is a maximum independent of G containing all the black vertices of G . \square

Theorem 4.4. *Every largest matching in a graph G constructed by Procedure 4.1 covers all the red vertices.*

Proof. We prove this by induction on the number of red vertices.

Base Case: Result is true when the number of red vertices is zero.

Induction Hypothesis: Every largest matching covers all the red vertices when the number of red vertices is $\ell - 1$.

Induction Step: Number of red vertices is ℓ .

Let the most recently added red vertex be v . The only descendants of v are black leaves, w_1, w_2, \dots, w_k , $k \geq 1$. Let M be a maximum matching in G . It may be verified that $G' = G - \{v, w_1, \dots, w_k\}$ can be obtained by Procedure 4.1 (it can be realised as a subsequence of Steps that produced G) and the number of red vertices in G' is $\ell - 1$.

Let $M' = M \cap E(G')$. Observe that $|M'| \geq |M| - 1$. Now M' is a maximum matching in G' , otherwise there exists a matching M'' , with $|M''| > |M'|$ in G' . Then $M'' \cup \{vw_1\}$ is a matching in G larger than M , a contradiction. By induction hypothesis, M' covers all the $\ell - 1$ red vertices of G' . Now, M' cannot be a maximum matching of G as $M' \cup \{vw_1\}$ is a bigger matching in G . Thus, $|M| - |M'| = 1$ and M covers all the red vertices in G . \square

Corollary 4.5. *If a graph G constructed by Procedure 4.1 is unicyclic non-KE, then after applying Step (4) it remains unicyclic non-KE.*

Proof. Let G' be the graph obtained from G after an application of Step (4) with a new black leaf u_1 attached to a red vertex u . Since u is matched by all maximum matchings in G , $\mu(G') = \mu(G)$. Also $\alpha(G') \leq \alpha(G) + 1$ (it can be shown that equality holds). Thus G' is unicyclic non-KE. \square

Theorem 4.6. *A graph obtained by Procedure 4.1 is a connected unicyclic non-KE graph.*

Proof. Follows from Lemma 4.2 and Corollary 4.5 and the fact that an odd cycle is a non-KE graph. \square

Now we shall show that all connected unicyclic non-KE graph G can be obtained by Procedure 4.1. The notation C is continued to denote the unique cycle in G . Consider the forest F obtained from G by deleting the edges in the cycle C . Define the vertices in the cycle as the roots of the trees in this forest. Let T_v be the component of the forest F rooted at v . The root v is not considered a leaf even if degree of v in T_v is 1. T_v is called nontrivial if it has more than one vertex.

Lemma 4.7. *If G is a unicyclic non-KE graph, then G does not have a leaf attached to the cycle C .*

Proof. Suppose there exists a leaf v attached to a vertex w in C . Let $G' = G - v - w$. Then G' is a forest (hence KE) and thus $\alpha(G') + \mu(G') = |V(G')| = |V(G)| - 2$.

Now if I is a maximum independent set of G' , then $I \cup v$ is independent of G , and if M is a maximum matching in G' , then $M \cup vw$ is a matching in G . Thus $\alpha(G) \geq |I \cup v| = |I| + 1 = \alpha(G') + 1$, and $\mu(G) \geq |M \cup \{vw\}| = |M| + 1 = \mu(G') + 1$. Hence $\alpha(G) + \mu(G) \geq \alpha(G') + \mu(G') + 2 = |V(G)|$, implying G is a KE graph, a contradiction. \square

Lemma 4.8. *Every nontrivial T_v contains a non-root vertex x with one of the following properties:*

1. x is the parent of more than one leaves.
2. x is the parent of only one leaf and the degree of x is 2.

Proof. Suppose T_v does not contain any non-root vertex with Property (1). Then T_v must contain a vertex with Property (2). Suppose not, then each leaf has a parent of degree more than 2 and the parent vertex does not have another leaf as a child. Let L be the set of leaves in T_v . From Lemma 4.7 it follows that $v \notin N(L)$. Let $P = V(T_v) - (L \cup N(L) \cup v)$. Clearly $V(T_v) = L \sqcup N(L) \sqcup P \sqcup v$ (\sqcup is used to denote disjoint union). It follows from the assumption that $|N(L)| = |L|$ and for all $x \in N(L)$, $\deg(x) \geq 3$. Also, $\deg(v) \geq 1$, $\deg(z) \geq 2$ for all $z \in P$, and $\deg(y) = 1$ for all $y \in L$. Hence $2|E(T_v)| = \text{sum of the vertex degrees in } T_v \geq$

$$|L| + 3|N(L)| + 2|P| + 1 = 4|L| + 2|P| + 1.$$

Thus, $|E(T_v)| > 2|L| + |P|$. But $|V(T_v)| = |L| + |N(L)| + |P| + 1$ implies

$$|E(T_v)| = |L| + |N(L)| + |P| = 2|L| + |P|, \text{ a contradiction.}$$

\square

Theorem 4.9. *Any connected unicyclic non-KE graph G can be constructed by Procedure 4.1.*

Proof. Let G be a minimal connected unicyclic non-KE graph that cannot be constructed by Procedure 4.1. Then G contains a nontrivial T_v and a vertex x with one of the two properties listed in Lemma 4.8.

Case 1: x satisfies Property (1).

Let x be the parent of leaves, say $y_1, y_2, \dots, y_k, k \geq 2$. Note that all the leaves y_1, y_2, \dots, y_k belong to any maximum independent set of G . Consider $G' = G - y_k$. Then $\alpha(G') = \alpha(G) - 1$ and $\mu(G') \leq \mu(G)$ (in fact equality holds).

$$\text{This implies, } \alpha(G') + \mu(G') \leq \alpha(G) - 1 + \mu(G) < |V(G)| - 1 = |V(G')|.$$

Thus, G' is a connected unicyclic non-KE subgraph of G . Minimality of G implies G' can be constructed by Procedure 4.1. But G can be obtained from G' by applying Step (4) and hence G can also be constructed by Procedure 4.1, a contradiction.

Case 2: x satisfies Property (2).

Let the unique child of x be the leaf y . Note that if I is a maximum independent set of G , then either $x \in I$ or $y \in I$ (but not both). Also if z is the parent of x and M is a maximum

matching in G , then either $zx \in M$ or $xy \in M$, but not both. Consider $G' = G - x - y$. Then $\alpha(G') \leq \alpha(G) - 1$ and $\mu(G') \leq \mu(G) - 1$.

This implies $\alpha(G') + \mu(G') \leq \alpha(G) - 1 + \mu(G) - 1 < |V(G)| - 2 = |V(G')|$.

Thus, G' is a connected unicyclic non-KE subgraph of G . Minimality of G implies G' can be constructed by Procedure 4.1. But G can be obtained from G' by applying Step (3) and hence G can also be constructed by Procedure 4.1, a contradiction. \square

Theorem 4.10. *For any connected unicyclic non-KE graph G the vertex colouring generated by Procedure 4.1 is independent of the particular sequence of Steps.*

Proof of this theorem is omitted. Reduction steps similar to the ones used in the proof of Theorem 4.9 may be used here as for each reduction step the choice of colour(s) for the deleted vertex (vertices) is unique. It may be noted that though the colouring is unique the same graph G can be generated by different sequence of steps.

Let G be a connected unicyclic non-KE graph with the odd cycle C . For the rest of this Section we assume C is of length $2m + 1$. Define B and R to be the set of black and red vertices of G , respectively.

Theorem 4.11. *For any connected unicyclic non-KE graph G , $\text{Core}(G) \subset B$ and $\alpha(G) = |B| + m$.*

Proof. By Theorem 4.3 there exists a maximum independent set I of G containing B . $I' := I - B \subset V(C)$ is a maximum independent set of C . Thus $|I'| = m$. For any $x \in V(C)$ there exists a maximum independent set I_x of C such that $x \notin I_x$. Then $B \cup I_x$ is a maximum independent of G . Hence, $x \notin \text{Core}(G)$ for any $x \in V(C)$. Thus, $V(C) \cap \text{Core}(G) = \emptyset$ and $\text{Core}(G) \subset B$. Also $\alpha(G) = |I| = |B \cup I'| = |B| + m$. \square

An edge is called red-black if one endpoint of the edge is a red vertex and the other endpoint is a black vertex.

Lemma 4.12. *There is a maximum matching M in any connected unicyclic non-KE graph G such that R is covered by red-black edges only.*

Proof. The lemma is proved by induction on $|R|$.

Base Case: $|R| = 0$. Then it is trivially true.

Induction Hypothesis: Let the assertion be true for $|R| = \ell - 1$.

Induction Step: Let $|R| = \ell$. Let v be the last red vertex added in G by Procedure 4.1. Then the only descendants of v are black leaves $w_1, \dots, w_k, k \geq 1$. Then $G' = G - \{v, w, \dots, w_k\}$ is again a unicyclic non-KE graph with the number of red vertices $\ell - 1$. By induction hypothesis, G' has a maximum matching M which covers all the red vertices by red-black edges only. Then $M \cup \{vw_1\}$ is a maximum matching in G . Since, vw_1 is a red-black edge the lemma is proved. \square

Corollary 4.13. *For any connected unicyclic non-KE graph G , $\mu(G) = |R| + m$.*

Proof. Let M be a maximum matching in G that covers all the red vertices by red-black edges only. If $e \in M$ is not a red-black edge then $e \in E(C)$. Thus number of non-red-black edges in M is the size of a maximum matching in C , which is m . Thus, $\mu(G) = |R| + m$. \square

Theorem 4.14. *B is a critical set in any connected unicyclic non-KE graph G .*

Proof. Note that $\text{Ker}(G) \subset \text{Core}(G) \subset B$ (Theorem 2.2 and Theorem 4.11). Also, $N(B) = R$ and by Lemma 4.12, there is a matching from R into B . Thus by Theorem 2.14, B is critical. \square

Corollary 4.15. *For any connected unicyclic non-KE graph G , $d_c(G) = |B| - |R| = \alpha(G) - \mu(G)$.*

Proof. Follows from Theorem 4.11, Corollary 4.13 and Theorem 4.14. \square

If G is a disconnected unicyclic non-KE graph, then $G = G' \oplus F$, (\oplus is used to denote disjoint union of graphs), where G' is the connected component of G containing the unique (odd) cycle and F is a forest. Observe that G' is a connected unicyclic non-KE graph. Conversely, if G' is any connected unicyclic non-KE graph and F is an arbitrary forest, then $G = G' \oplus F$ is a unicyclic non-KE graph.

By the corollary above, $d_c(G') = \alpha(G') - \mu(G')$. It is known that for KE graphs the critical difference equals the difference between independence number and matching number [6]. Hence $d_c(F) = \alpha(F) - \mu(F)$. It may be verified that $d_c(G) = d_c(G') + d_c(F)$, $\alpha(G) = \alpha(G') + \alpha(F)$ and $\mu(G) = \mu(G') + \mu(F)$. Thus $d_c(G) = \alpha(G') - \mu(G') + \alpha(F) - \mu(F) = \alpha(G) - \mu(G)$. Hence $d_c(G) = \alpha(G) - \mu(G)$ holds for any disconnected unicyclic non-KE graph G too and thus Conjecture 1.3 is proved.

Note that in the Procedure 4.1, if G doesn't contain a red vertex (i.e., Step (3) is never applied and hence $G = C$), applying Step (4) doesn't change G . We call such a Step (4) trivial.

Corollary 4.16. *Critical difference of a connected unicyclic non-KE graph G is the number of times a nontrivial Step (4) is applied in the construction of G by Procedure 4.1.*

Proof. As the red vertices in the construction of a graph G are in a one-to-one correspondence with the black vertices added by Step (3), the result follows. \square

This also shows that for any connected unicyclic non-KE graph G the number of times each Step (excluding any trivial Step (4)) is chosen in Procedure 4.1 is fixed even though the order may not be unique.

5 Concluding Observations

Ker , Core , Diadem and other related notions have been well studied. Some basic questions still remain. Two of the least understood problems are:

Problem 5.1. [3, 9] *Characterise graphs such that $\text{Core}(G)$ is critical.*

Problem 5.2. [3, 9] *Characterise graphs with $\text{Ker}(G) = \text{Core}(G)$.*

We tried to use Edmonds-Gallai decomposition to understand $\text{Ker}(G)$ better. The observations in this Section seem to be insufficient to address the above problems but may be useful in further work.

For the completeness and to fix the notation Edmonds-Gallai decomposition is stated below.

Let \mathcal{D} be the set of vertices not covered (*missed*) by a largest matching in G . Let \mathcal{A} be the set of neighbours of \mathcal{D} not in \mathcal{D} . The set \mathcal{C} contains the remaining vertices. Edmonds-Gallai Decomposition of G is the partition of $V(G)$ into the three sets $\mathcal{A}, \mathcal{C}, \mathcal{D}$.

Theorem 5.3 (Edmonds-Gallai Structure Theorem). *Let \mathcal{A} , \mathcal{C} and \mathcal{D} be the sets in the Edmonds-Gallai Decomposition of a graph G . Let G_1, \dots, G_k be the connected components of $G[\mathcal{D}]$. If M is a maximum matching in G , then the following properties hold.*

1. *All the vertices in \mathcal{C} are matched amongst themselves.*
2. *Every vertex in \mathcal{A} is matched to distinct components in $G[\mathcal{D}]$.*
3. *Each of G_1, \dots, G_k is factor critical (i.e., admits a perfect matching after deleting any one of the vertices).*
4. *If $\emptyset \neq S \subset \mathcal{A}$, then $N(S)$ has a vertex in at least $|S| + 1$ of G_1, \dots, G_k .*

First a simple observation is stated.

Lemma 5.4. *Let G be a disjoint union of factor critical graphs each of order strictly greater than 1. If $S \in \text{Ind}(G)$ and nonempty then $d(S) < 0$.*

Proof. Without loss of generality it may be assumed that G is connected. Choose a vertex $v \in N(S)$. Since, G is a factor critical graph, $G - v$ has a perfect matching. As S is an independent set of $G - v$, $|N(S) - v| \geq |S|$. Therefore $d(S) = |S| - |N(S)| = |S| - |N(S) - v| - 1 < 0$. \square

Theorem 5.5. *Let S be a critical independent set in G and G_1, G_2, \dots, G_k be the components of $G[\mathcal{D}]$. Then*

$$S \subset \mathcal{C} \cup \left(\bigsqcup_{\substack{i=1 \\ |V(G_i)|=1}}^k V(G_i) \right).$$

Proof. Let $Y = \mathcal{C} \cup \left(\bigsqcup_{\substack{i=1 \\ |V(G_i)|=1}}^k V(G_i) \right)$.

$$\text{Let } X = S \cap \overline{Y} = S \cap (\mathcal{A} \cup \left(\bigsqcup_{\substack{i=1 \\ |V(G_i)|>1}}^k V(G_i) \right)).$$

It will be shown that $X = \emptyset$. So assume $X \neq \emptyset$.

Assertion: $|X| - |N(X) \cap \mathcal{D}| < 0$.

Let $\mathcal{J} = \{i \in [k] : |V(G_i)| > 1 \text{ and } V(G_i) \cap S \neq \emptyset\}$ and $m = |\mathcal{J}|$.

Let $X_i = S \cap V(G_i)$, for $i \in \mathcal{J}$ and $X_0 = S \cap \mathcal{A}$. Then, $X = X_0 \sqcup (\sqcup_{i \in \mathcal{J}} X_i)$. As G_i is factor critical, by Lemma 5.4, $|N(X_i) \cap V(G_i)| \geq |X_i| + 1$, for $i \in \mathcal{J}$.

$$\text{Thus } \sum_{i \in \mathcal{J}} |N(X_i) \cap V(G_i)| \geq \sum_{i \in \mathcal{J}} |X_i| + m.$$

Case 1: $X_0 = \emptyset$.

$$\begin{aligned} \text{Since in this case } m > 0, \quad & \sum_{i \in \mathcal{J}} |N(X_i) \cap V(G_i)| > \sum_{i \in \mathcal{J}} |X_i| = |X| \\ \Rightarrow \quad & |N(X) \cap \mathcal{D}| = |(\cup_{i \in \mathcal{J}} N(X_i)) \cap \mathcal{D}| > |X|. \end{aligned}$$

Case 2: $X_0 \neq \emptyset$.

Let $W = \sqcup_{i \in \mathcal{J}} V(G_i)$. It will be shown that $|N(X_0) \cap (\mathcal{D} - W)| > |X_0| - m$ when $|X_0| \geq m$. By Edmonds-Gallai Structure Theorem, X_0 has neighbours in at least $|X_0| + 1$ components of $G[\mathcal{D}]$. Thus, X_0 has neighbours in at least $|X_0| + 1 - m$ many components of $G[\mathcal{D}]$ different from G_i 's, $i \in \mathcal{J}$. Thus, $|N(X_0) \cap (\mathcal{D} - W)| \geq |X_0| + 1 - m > |X_0| - m$.

$$\text{Also } |(\cup_{i \in \mathcal{J}} N(X_i)) \cap W| = \sum_{i \in \mathcal{J}} |N(X_i) \cap V(G_i)| \geq \sum_{i \in \mathcal{J}} |X_i| + m.$$

$$\begin{aligned} \text{Hence, } |N(X) \cap \mathcal{D}| &\geq |N(\cup_{i \in \mathcal{J}} X_i) \cap W| + |N(X_0) \cap (\mathcal{D} - W)| \\ &> \sum_{i \in \mathcal{J}} |X_i| + m + |X_0| - m \\ &= |X|. \end{aligned}$$

Thus, in both the cases $|N(X) \cap \mathcal{D}| > |X|$ and the Assertion is proved.

Note that, $N(S - X) \subset N(S) - (N(S) \cap \mathcal{D})$. Thus $|N(S - X)| \leq |N(S)| - |N(S) \cap \mathcal{D}| = |N(S)| - |N(X) \cap \mathcal{D}|$. From these observations $d(S - X)$ can be computed as:

$$\begin{aligned} d(S - X) &= |S| - |X| - |N(S - X)| \\ &\geq |S| - |X| - |N(S)| + |N(X) \cap \mathcal{D}| \\ &= d(S) + |N(X) \cap \mathcal{D}| - |X| \\ &> d(S). \end{aligned}$$

This contradicts the criticality of S . □

Corollary 5.6. *Let G_1, G_2, \dots, G_k be the components of $G[\mathcal{D}]$. Then*

$$\text{Ker}(G) \subset \bigsqcup_{\substack{i=1 \\ |V(G_i)|=1}}^k V(G_i).$$

Proof. Since $\text{Ker}(G)$ is critical and independent it is enough to show that $X := \text{Ker}(G) \cap \mathcal{C} = \emptyset$. So assume $X \neq \emptyset$. By Edmonds-Gallai Structure Theorem, vertices in \mathcal{C} are matched amongst themselves by any maximum matching. So there is a matching from X into $N(X) \cap \mathcal{C}$, which implies $|N(X) \cap \mathcal{C}| \geq |X|$. Observe that $N(\text{Ker}(G) - X) \subset \mathcal{A}$, and hence $N(\text{Ker}(G) - X) \subset N(\text{Ker}(G)) - (N(X) \cap \mathcal{C})$.

$$\begin{aligned} \text{Then, } d(\text{Ker}(G) - X) &= |\text{Ker}(G)| - |X| - |N(\text{Ker}(G) - X)| \\ &\geq |\text{Ker}(G)| - |X| - |N(\text{Ker}(G))| + |N(X) \cap \mathcal{C}| \\ &\geq d_c(G), \end{aligned}$$

a contradiction to the minimality of $\text{Ker}(G)$ as a critical independent set. □

It would also be nice to generalise the results proved for unicyclic non-KE graphs to graphs G for which $\alpha(G) + \mu(G) = |V(G)| - k$, where k is a constant. It would be interesting to look at properties of Ker, Core, Diadem etc. for graphs that are “close” to bipartite graphs.

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